Simplifying Problems with Canonical Trafos – the HO again:

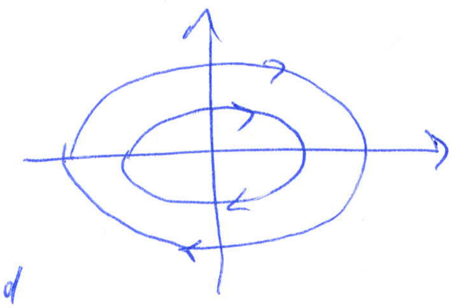
$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 q^2$$

$$\Rightarrow \dot p = -m\omega^2 q \quad , \quad \dot q = p/m$$

$$\rightarrow q(t) = A\cos(\omega(t-t_0)) \quad , \quad p = -m\omega A \sin(\omega(t-t_0)) \, ,$$

with A, t_0 integration constants.

Flows in phase space are <u>ellipses</u>:



Make a canonical trafo & mix up q's and p's:

$$(q,p) \rightarrow (\theta, I)$$

new "position" (θ) new "momentum" (I)

$$q = \sqrt{\frac{2I}{m\omega}}\sin\theta \quad , \quad p = \sqrt{2Im\omega}\cos\theta$$

(looks a bit odd, but will be useful → discuss complex version in ~~lecture~~ problem to highlight parallels to treatment in QM)

1) Check poisson brackets: It's easiest to check

$$\{q,p\}_{(\theta,I)} = \frac{\partial q}{\partial\theta}\frac{\partial p}{\partial I} - \frac{\partial p}{\partial \theta}\frac{\partial q}{\partial I}$$

$$= \underbrace{\sqrt{\frac{2}{m\omega}}\cdot\sqrt{2m\omega}}_{=2}\left\{\sqrt{I}\sin\theta, \sqrt{I}\cos\theta\right\}$$

$$= 2\left(\left(\sqrt{I}\cos\theta\right)\left(\frac{1}{2\sqrt{I}}\cos\theta\right) - \left(-\sqrt{I}\sin\theta\right)\left(\frac{1}{2\sqrt{I}}\sin\theta\right)\right)$$

$$= \cos^2\theta + \sin^2\theta = \underline{1} \checkmark$$

(Check that $MJM^T = J$ as an exercise, where

$$M = \begin{pmatrix} \partial\theta/\partial q & \partial\theta/\partial p \\ \partial I/\partial q & \partial I/\partial p \end{pmatrix} = \begin{pmatrix} m\omega/p \cos^2\theta & -m\omega q/p^2 \cos^2\theta \\ m\omega q & p/m\omega \end{pmatrix} \Bigg)$$

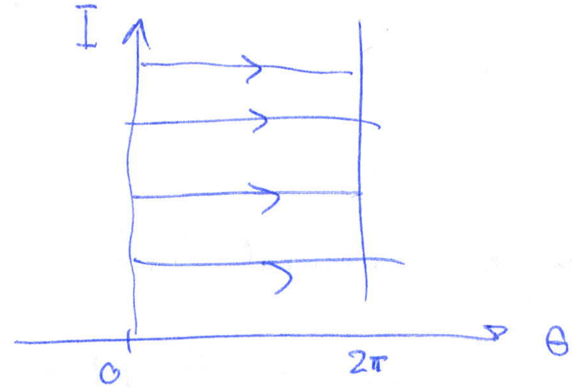
Now look at H in new variables:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = \frac{1}{2m}\left(2m\omega I\right)\sin^2\theta + \frac{1}{2} m\omega^2 \frac{2I}{m\omega}\cos^2\theta = \underline{\omega I} \, !$$

$$\Rightarrow \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega \, , \quad \dot{I} = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow I = \text{const.}$$

"coordinate" "momentum"

How does this look in phase space?



Checking dimension, we see that I has dimensions energy × time, like the action – thus, (θ, I) are referred to as action-angle variables.

↳ This "straightening out" of the flow can not be done ~~for integrable~~ for all systems. If it's possible, we have

$$(q_i, p_i) \rightarrow (\theta_j, I_j)$$

and $H = H(I_j)$, but no θ-dependence. Thus

$$\dot{\theta}_k = \frac{\partial H}{\partial I_k} = \omega_k(I_j) \quad \Rightarrow \quad \theta_k(t) = \omega_k t + \text{const.}$$

If such a trafo is possible, the system is integrable.

For bounded motion, the θ_j are customarily scaled to $0 \leq \theta_j \leq 2\pi$,

and (θ_j, I_j) are called action-angle variables!